Fast Quantum State Reconstruction via Accelerated Non-Convex Programming

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[QST objective]

- A quantum state can be represented by a density matrix ρ which is a complex, positive semi-definite (PSD) matrix with unit trace
- Goal of QST: estimate ρ, given the measurement data
- The density matrix of an n-qubit mixed state can be written as a mixture of r pure states:

$$\rho = \sum_k^r p_k \Psi_k \Psi_k^\dagger \in \mathbb{C}^{2^n \times 2^n}$$

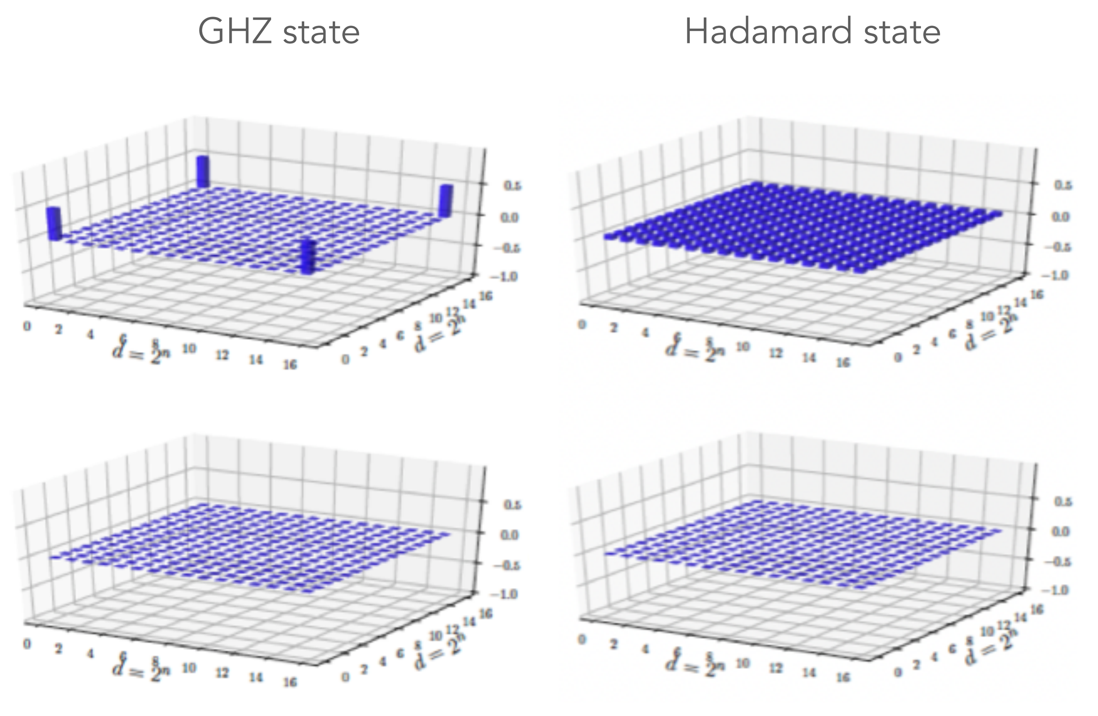
where p_k is the probability of finding ρ in the pure state Ψ_k.

- QST objective: estimation of a low-rank density matrix $\rho^\star \in \mathbb{C}^{d\times d}$ on an n-qubit Hilbert space ($d = 2^n$):

$$\min_{\rho \in \mathbb{C}^{d\times d}} \quad F(\rho) := \frac{1}{2m}\|\mathcal{A}(\rho) - y\|_2^2$$
$$\text{subject to} \quad \rho \succeq 0, \ \text{rank}(\rho) \le r$$

- $\mathcal{A} : \mathbb{C}^{2^n \times 2^n} \to \mathbb{R}^m$ is the linear sensing map such that $\mathcal{A}(\rho)_k = \text{Tr}(A_k \rho)$ for $k = 1, \ldots, m$ (the Born rule)

[Structured density matrices]



[Motivation of low-rank prior]

- Classically (without low-rank prior), the sample complexity m for reconstructing $\rho^\star \in \mathbb{C}^{d\times d}$ is $O(d^2)$
- [Gross et al., 2010] proved that a rank-r density matrix can be reconstructed with $m = O(r \cdot d \cdot \text{poly}\log(d))$ measurements instead
- Many density matrices have low-rank structure
- Caveat: low-rankness is a non-convex constraint, which is tricky to handle

[Modified QST objective]

- By rewriting $\rho = UU^\dagger$, both the PSD and the low-rank constraints are automatically satisfied, leading to the following unconstrained non-convex formulation:

$$\min_{U \in \mathbb{C}^{d\times r}} \ G(U) := F(UU^\dagger) = \frac{1}{2m}\|\mathcal{A}(UU^\dagger) - y\|_2^2.$$

- Expensive projection operators are no longer needed

[Algorithms]

Factored Gradient Descent [Kyrillidis et al., 2019]

$$U_{k+1} = U_k - \eta \nabla f(U_k U_k^\dagger) \cdot U_k$$
$$= U_k - \eta \mathcal{A}^\dagger \left(\mathcal{A}(U_k U_k^\dagger) - y\right) \cdot U_k$$

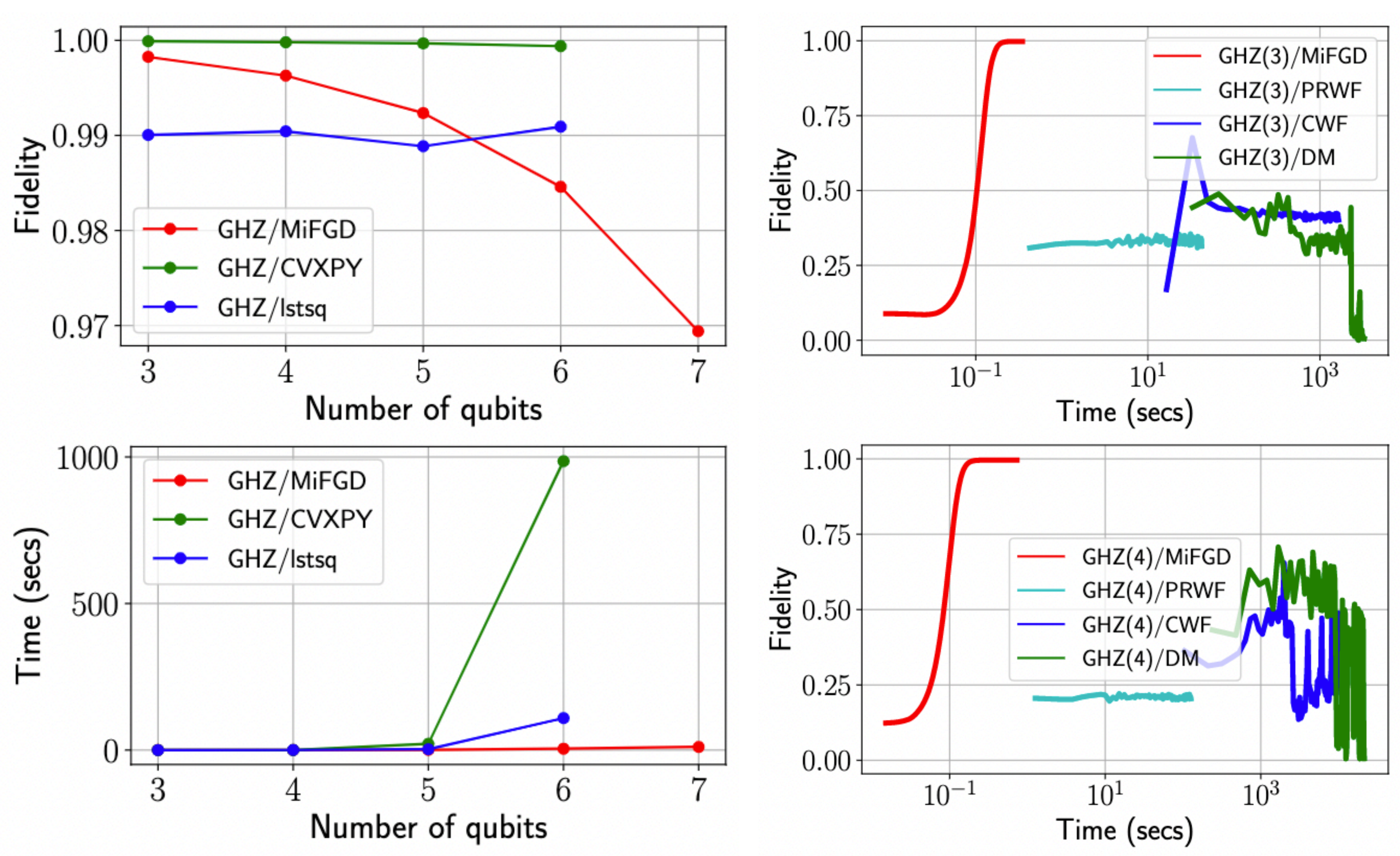
Momentum-inspired FGD

$$U_{k+1} = Z_k - \eta \mathcal{A}^\dagger \left(\mathcal{A}(Z_k Z_k^\dagger) - y\right) \cdot Z_k$$
$$Z_{k+1} = U_{k+1} + \mu\left(U_{k+1} - U_k\right)$$

Theorem: Assume that $\mathcal{A}$ satisfies the RIP with constant $\delta_{2r} \le 1/10$. Initialize $U_0 = U_{-1}$ such that $\min_{R\in\mathcal{O}} \|U_0 - U^\star R\|_F \le \frac{\sqrt{\sigma_r(\rho^\star)}}{10^3\sqrt{\kappa\tau(\rho^\star)}}$, where $\kappa := \frac{1+\delta_{2r}}{1-\delta_{2r}}$ is the (inverse) condition number of $\mathcal{A}$, $\tau(\rho) := \sigma_1(\rho)/\sigma_r(\rho)$ is the condition number of ρ with $\text{rank}(\rho) = r$, and $\sigma_i(\rho)$ is the i-th singular value of ρ. Set the step size η such that $\left[1 - \left(\frac{\sqrt{1+\delta_{2r}} - \sqrt{1-\delta_{2r}}}{(\sqrt{2}+1)\sqrt{1+\delta_{2r}}}\right)^4\right] \cdot \frac{10}{4\sigma_r(\rho^\star)(1-\delta_{2r})} \le \eta \le \frac{10}{4\sigma_r(\rho^\star)(1-\delta_{2r})}$, and the momentum parameter $\mu = \frac{\varepsilon_\mu}{2\cdot 10^3 r\tau(\rho^\star)\sqrt{\kappa}}$, for user-defined $\varepsilon_\mu \in (0,1]$. Then, for the (noiseless) measurement data $y = \mathcal{A}(\rho^\star)$ with $\text{rank}(\rho^\star) = r$, the output of the MiFGD satisfies the following: for any $\epsilon > 0$, there exist constants C_ϵ and $\tilde{C}_\epsilon$ such that, for all k,

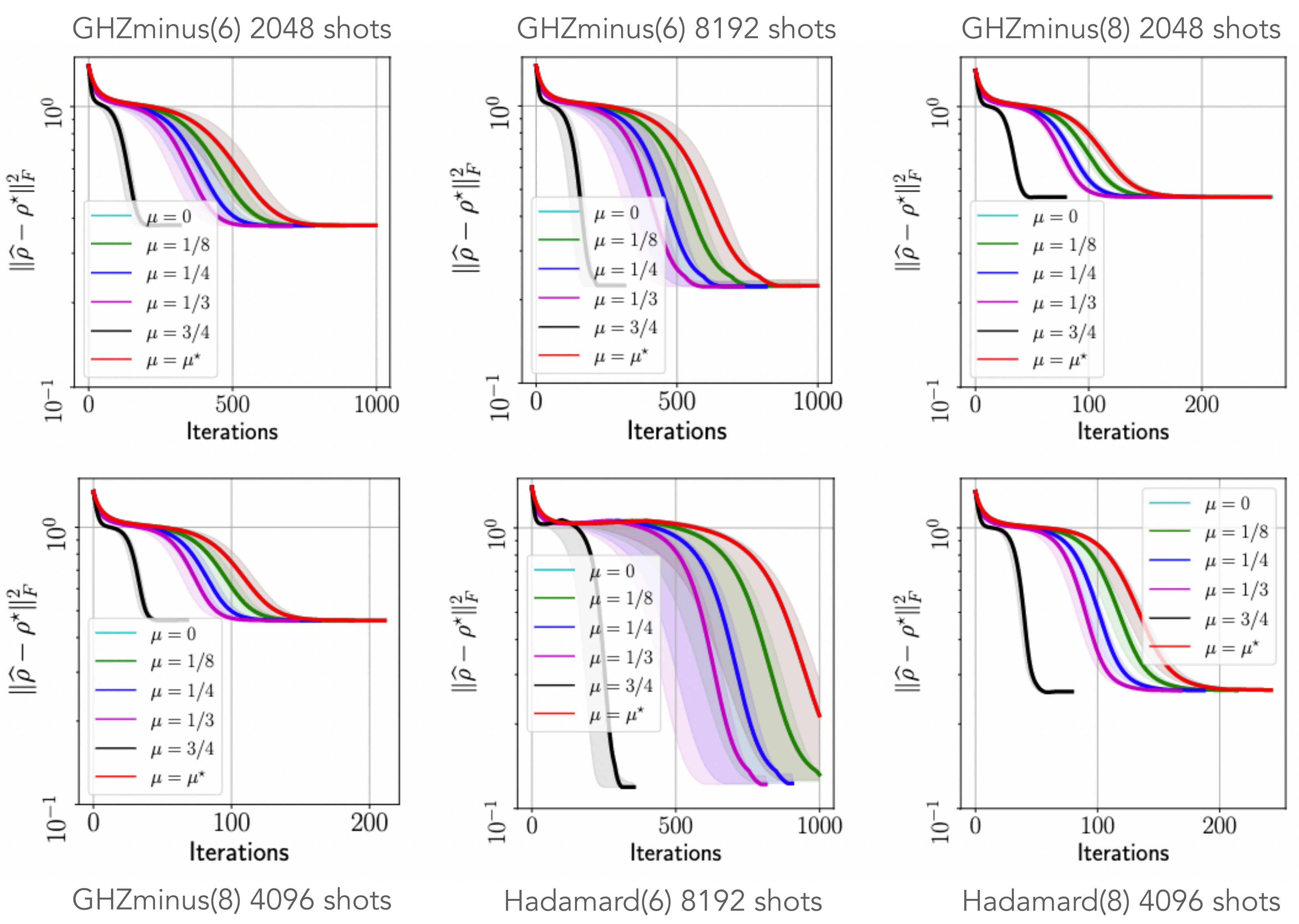
$$\left(\min_{R\in\mathcal{O}} \|U_{k+1} - U^\star R\|_F^2 + \min_{R\in\mathcal{O}} \|U_k - U^\star R\|_F^2\right)^{1/2} \le C_\epsilon \left(1 - \sqrt{\frac{1-\delta_{2r}}{1+\delta_{2r}}} + \epsilon\right)^{k+1} \min_{R\in\mathcal{O}} \|U_0 - U^\star R\|_F + O(\mu).$$

[Comparison with Qiskit and Cucumber (DNN)]



- Extensive additional experiments provided in the paper
- Hadamard state, random state, and higher number of qubits

[MiFGD performance (real quantum data)]



- Comparison of simulated data VS. real quantum data
- Open source compatible with Qiskit: github.com/gidiko/MiFGD